Technical Report 202

BEAM TEMPERATURE EFFECTS ON GROWTH RATE OF ELECTROSTATIC INSTABILITY FOR AN ELECTRON BEAM PENETRATING A PLASMA

Prepared for:

OFFICE OF NAVAL RESEARCH
WASHINGTON 25, D.C.

CONTRACT Nonr-2728(00)

By: *H. E. Singhaus*

STANFORD RESEARCH INSTITUTE

MENLO PARK, CALIFORNIA

*SRI

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November 1963

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Approved:

CARSON FLAMMER, MANAGER MATHEMATICAL PHYSICS DEPARTMENT

A. M. PETERSON, ASSISTANT DIRECTOR ELECTRONICS AND RADIO SCIENCES DIVISION

Copy No. 73

SUMMARY

The electrostatic instability of an infinite electron beam penetrating an infinite plasma is analyzed. The objective is to determine the effect of beam temperature upon the growth rate of the electrostatic waves (a Gaussian velocity distribution is assumed). In addition the role of collisions within the plasma is explored.

A conclusion of this study is that a high temperature limit for the beam can be defined in terms of the plasma frequencies of the beam and of the plasma. At the high temperature limit, the plasma collisions can quench the electrostatic instability; at lower temperatures, collisions tend to enhance the growth rate.

The results presented here are based upon a numerical analysis of the usual dispersion equation that is obtained for this beam-plasma system; the numerical work was carried out with the aid of a Burroughs 220 computer with a program written in BALGOL. In addition to the numerical results, approximate analytic expressions for the growth rate are derived for a number of cases of interest.

CONTENTS

ILLUSTRATIONS

I INTRODUCTION

This report presents the results of a numerical investigation of the electrostatic instability of a system that consists of a relativistic electron beam penetrating a plasma. Both the beam and plasma are assumed to be uniform and infinite in extent. The plasma is assumed to be dense enough that collisions between plasma electrons and other plasma particles may play an important role, but in other respects the plasma is assumed to be "cold"--that is, the plasma velocity distribution is ignored. On the other hand, the beam is allowed to have a finite velocity distribution--assumed here to be Gaussian--but all collisions between beam particles or between beam and plasma particles are ignored.

The physical situation described above has been considered already in a number of papers.[1-3*] The usual treatment provides a dispersion relation for the response of the system to small-scale disturbances, but this dispersion equation does not appear to have been solved analytically, except in the limits of very high or very low beam temperature. Here some new analytic solutions of the dispersion equation are given for the high and intermediate temperature range, but these results also do not cover the full range of beam temperature. Therefore, in order to explore more fully the effects of beam temperature upon the growth rate of an initial disturbance, we have resorted to numerical methods. In that part of the beam temperature range where approximate analytic solutions exist, the numerical and analytic results agree quite well.

It is found that one can obtain from the dispersion equation a simple criterion for a "high temperature" beam: such a beam is one in which low-frequency oscillations are quenched, so that, for a sufficiently high beam temperature, growing waves can exist only at frequencies very near to the plasma frequency. This conclusion is substantiated by the numerical work.

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*References are listed at the end of the report.

In addition to the effects of beam temperature, the role of the "effective" collision frequency of the plasma particles is also explored in the numerical work reported here. It is found that at low beam temperature, collisions can enhance the growth rate of the low-frequency oscillations. On the other hand, in the high temperature limit, the high-frequency oscillations can be quenched by a sufficiently high collision frequency. An approximate expression for the minimum collision frequency required to quench the high-frequency instability can be derived from the dispersion equation; the numerical work indicates that this derived expression is a reasonably good approximation.

II THE DISPERSION EQUATION FOR ELECTROSTATIC WAVES

The dispersion equation with which we are concerned has been derived elsewhere.[1-3] The following assumptions and approximations are intrinsic to these derivations:

(1) In the steady state the plasma is assumed to be uniform and infinite in extent.

(2) In the steady state the electron beam is assumed to be uniform and infinite. However, it is usually assumed that the same dispersion equation holds to a good approximation for a uniform beam of finite cross-section, provided that the wavelength of the instability is small compared to the beam radius.

(3) Electron collisions within the plasma are taken into consideration, but other aspects of the plasma velocity distribution are ignored (the "cold" plasma approximation).

(4) All steady-state magnetic fields are ignored, including the self-field of the beam.

(5) Perturbations are considered in the linear approximation, and in the form of a single Fourier component, $\exp(i\,\vec{k}\cdot\vec{r} - i\omega t)$. The wave number, $\vec{k}$, is assumed to be real, so that any solution of the dispersion equation with frequency ω having a positive imaginary part indicates growing solutions.

(6) The beam density is assumed to be small compared with the density of charged particles in the plasma.

Under the foregoing approximations, and assuming that the beam has a finite Gaussian velocity distribution as defined by the root-mean-square velocity difference, u, and the average beam velocity V_o, one can obtain a dispersion relation for the electrostatic waves of the form

$$1 = \frac{\omega_p^2}{\omega(\omega + i\nu)} + \frac{\omega_b^2}{\sqrt{2\pi}\, u^3} \int_{-\infty}^{\infty} \frac{v' \exp\left(-\frac{v'^2}{2u^2}\right) dv'}{k(\omega - kV_o - kv')} \tag{1}$$

where ν is the effective collision frequency of the plasma electrons. The plasma frequency, ω_p, and the beam plasma frequency, ω_b, are defined in rationalized MKS units by

$$\omega_p^2 = \frac{n_p e^2}{\epsilon_o m} \text{ and } \omega_b^2 = \frac{n_b e^2}{\epsilon_o \gamma^3 m} \tag{2}$$

with

$$\gamma = \left(1 - V_o^2/c^2\right)^{-\frac{1}{2}} \quad . \tag{3}$$

According to the formulation of Bludman, Watson, and Rosenbluth,[1] an equation similar to Eq. (1) is obtained for the case of electrostatic waves that are in the beam direction;* for waves at an arbitrary angle to the beam, a more involved expression is obtained, which requires integration over the transverse velocity structure of the beam. On the other hand, in the formulation by Ascoli,[2] an equation similar to Eq. (1) is obtained for waves at arbitrary angle to the beam, provided that V_o and u are interpreted as the average beam velocity and beam velocity spread, respectively, in the direction of the wave.

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*See Eqs. (4.10) and (4.11) on p. 754 of Ref. 1. Equation (1) can be obtained from these equations after some algebraic manipulations for the case of longitudinal waves ($k_x = 0$).

For the numerical work it is convenient to introduce the new variables

$$t = \frac{v'}{\sqrt{2}\, u} \text{ and } z = \frac{\omega - kV_o}{\sqrt{2}\, ku} \tag{4}$$

so that Eq. (1) can be written

$$1 = \frac{\omega_p^2}{\omega(\omega + i\nu)} - \frac{1}{\sqrt{\pi}} \frac{\omega_b^2}{k^2 u^2} \int_{-\infty}^{\infty} \frac{t\, e^{-t^2}\, dt}{t - z} \quad . \tag{5}$$

Now writing $\frac{t}{t - z} = 1 + \frac{z}{t - z}$, it is readily shown that

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t\, e^{-t^2}}{t - z}\, dt = 1 + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}\, dt}{t - z} \quad . \tag{6}$$

The definition

$$Z(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}\, dt}{t - z} \tag{7}$$

and Eq. (6) can be used to rewrite Eq. (5) in the form

$$1 = \frac{\omega_p^2}{\omega(\omega + i\nu)} - \frac{\omega_b^2}{u^2 k^2} [1 + zZ] \quad . \tag{8}$$

The integral Z(z) and its derivative have been published[4] in tabular form for values of the argument $z = x + iy$ in the range $0 \leq x \leq 10$ and $-10 \leq y \leq 10$. However, for work reported here, it was found advisable to calculate Z(z) directly.

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III APPROXIMATE SOLUTIONS OF THE DISPERSION EQUATION

Although this paper is concerned primarily with numerical solutions of Eq. (8), the numerical procedure that is employed requires reasonably good first approximations to the solution in order to obtain convergence to that solution. In this section, some approximate analytic solutions of the dispersion relation are obtained. The nature of the analytic solutions depends to a large extent upon the beam temperature. It was found convenient for present purposes to define high- and low-temperature limits in terms of the system parameters. Thus, in the first part of this section, the high-temperature limit is defined, and approximate analytic solutions are obtained in this limit. Subsequently, the solutions at low and intermediate temperatures are considered.

The analytic solutions obtained in the limits of high and low temperatures are, in general, in quite close agreement with the numerical results. At intermediate temperatures the situation is different, in that good analytic solutions are obtained for part of this range only. In that part of the intermediate temperature range where analytic solutions are not obtained, numerical results are obtained more or less by trial and error.

A. The High-Temperature Limit

1. General

It is easy to show that for sufficiently high beam temperature, the infinite beam-plasma system cannot support growing waves at frequencies well below the plasma frequency. The important point to be made here is that this condition does not depend upon the beam temperature alone, but upon both the beam temperature and the ratio of plasma frequency to beam-plasma frequency. In order to establish this high-temperature criterion, let us define

$$Z(z) = X(z) + i\,Y(z) \tag{9}$$

and

$$
\left.\begin{aligned} z &= x + iy \\ \omega &= \omega_1 + i\omega_2 \end{aligned}\right\} \quad . \tag{10}
$$

Then taking real and imaginary parts of Eq. (8), one obtains the two equations

$$
\omega_1 \left[1 - \frac{\omega_p^2}{\omega_1^2 + (\omega^2 + \nu)^2}\right] + \frac{\omega_b^2}{u^2k^2}\left[\omega_1(1 + xX - yY) - \omega_2(xY + yX)\right] = 0 \tag{11}
$$

$$
\omega_2 \left[1 + \frac{\left(1 + \frac{\nu}{\omega_2}\right)\omega_p^2}{\omega_1^2 + (\omega_2 + \nu)^2}\right] + \frac{\omega_b^2}{u^2k^2}\left[\omega_1(xY + yX) + \omega_2(1 + xX - yY)\right] = 0 \quad . \tag{12}
$$

Equation (11) can be rewritten

$$
xY + yX = \frac{\omega_1}{\omega_2}\left[(1 + xX - yY) + \frac{u^2k^2}{\omega_b^2}\left(1 - \frac{\omega_p^2}{\omega_1^2 + (\omega_2 + \nu)^2}\right)\right] \tag{13}
$$

so that from Eqs. (12) and (13)

$$
1 + \frac{\omega_b^2}{u^2k^2}(1 + xX - yY) + \frac{\omega_p^2}{\omega_1^2 + (\omega_2 + \nu)^2}\left[\frac{\omega_2^2 + \nu\omega_2 - \omega_1^2}{\omega_1^2 + \omega_2^2}\right] = 0 \quad . \tag{14}
$$

It is now assumed that

$$
\begin{aligned}
&(a) \quad \omega_1 \ll \omega_p \\
&(b) \quad 0 < \omega_2 \ll \omega_1 \\
&(c) \quad \nu \ll \omega_1 \\
&(d) \quad \omega_1 \approx kV_o
\end{aligned}
\tag{15}
$$

so that Eq. (14) can be rewritten, approximately,

$$1 + \frac{\omega_b^2}{u^2 k^2} (1 + xX - yY) = \frac{\omega_p^2}{\omega_1^2} \quad . \tag{16}$$

Furthermore, in view of the assumptions of (15-a) and (15-d)

$$1 + xX - yY \approx \frac{\omega_p^2}{\omega_b^2} \frac{u^2}{V_o^2} \quad . \tag{17}$$

We make use of the condition*

$$1 + xX - yY < 1, \text{ for } \omega_2 > 0 \quad . \tag{18}$$

Consequently, for a beam-plasma system such that the right-hand side of Eq. (17) is much greater than unity, Eq. (17) cannot be satisfied. We conclude that the low temperature instabilities that satisfy the dispersion equation and Eqs. (15) do not exist at beam temperatures high enough that

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*This condition is obtained by examination of the published results of Fried and Conte, Ref. 4. To facilitate use of this reference, note that $1 + xX - yY = -\frac{1}{2}\,\mathrm{Re}(Z')$.

$$\tau \equiv \left(\frac{\omega_p}{\omega_b} \frac{u}{V_o}\right) >> 1 \quad . \tag{19}$$

The results of our numerical investigation verify the validity of this criterion. When the condition imposed by Eq. (19) is satisfied, instabilities are found in only a very narrow range of frequencies in the vicinity of the plasma frequency, ω_p.

2. High-Frequency Instabilities and Collisional Damping in the High-Temperature Limit

At frequencies near the plasma frequency, ω_p, an unstable mode does exist in the high temperature limit. Consider now the question of collisional damping of this unstable mode. First, let us assume the conditions:

$$\begin{aligned} &\text{(a)} \quad \omega_1 \approx kV_o \approx \omega_p \\ &\text{(b)} \quad \nu \ll \omega_1 \end{aligned} \tag{20}$$

and look for conditions such that

$$\omega_2 \to 0 \quad . \tag{21}$$

According to the assumptions of Eqs. (20) and (21), Eq. (12) can be rewritten

$$\frac{\nu \, \omega_p^2}{\omega_1^2} + \frac{\omega_b^2}{u^2 k^2} \, \omega_1 x Y(x) \approx 0 \tag{22}$$

or

$$\nu \approx \frac{- \, \omega_b^2}{u^2 k^2 \omega_p^2} \, \omega_1^3 \, xY(x) \quad . \tag{23}$$

It can be shown that*

$$Y(x) = \sqrt{\pi}\, e^{-x^2} \tag{24}$$

so that Eq. (23) can be written

$$\nu = -\frac{\sqrt{\pi}\,\omega_b^2}{\omega_p^2\, u^2}\,\frac{\omega_1^3\, x e^{-x^2}}{k^2} \quad . \tag{25}$$

The right-hand side of this equation has a maximum when

$$\frac{\omega_1^3}{k^2}(1 - 2x^2) + \frac{3\omega_1^2}{k^2}(\omega_1 - kV_o) = 0 \quad . \tag{26}$$

According to assumption (20-a), the last term of Eq. (26) can be neglected. Then, approximately

$$x = -\frac{1}{\sqrt{2}} \quad , \tag{27}$$

where the negative root is chosen in order to satisfy Eq. (22). Thus, to damp the fastest growing instability we require a collision rate**

$$\nu \gtrsim \sqrt{\frac{\pi}{2e}}\,\frac{\omega_p}{\left(\frac{\omega_p}{\omega_b}\frac{u}{V_o}\right)^2} \quad . \tag{28}$$

---

*See, for example, Fried and Conte, Ref. 4. Note that the term Y(x), as defined by Eq. (9), is not the same as Y(x) defined on p. 3 of Ref. 4.

**This result agrees with Ascoli (Ref. 2), but the authors of Ref. 1 obtain a somewhat different result (see note added in proof on p. 757 of Ref. 1).

The numerical work verifies that Eq. (28) provides a quite good estimate of the minimum collision rate for damping.

Now consider the instability growth rate when the collision rate is much less than the damping rate of Eq. (28). Let us assume the conditions

$$\begin{aligned} &\text{(a)} \quad \omega_1 \approx kV_o \approx \omega_p \\ &\text{(b)} \quad \nu \ll \omega_2 \ll \omega_1 \quad . \end{aligned} \tag{29}$$

From Eq. (11) we get the approximate condition

$$1 + xX - yY = \frac{\omega_2}{\omega_1} (xY + yX) \quad . \tag{30}$$

Using Eq. (30) in Eq. (12),

$$2\omega_2 \approx - \frac{\omega_b^2}{u^2k^2} (xY + yX) \left(\omega_1 + \frac{\omega_2^2}{\omega_1}\right) \tag{31}$$

or

$$\omega_2 \approx - \frac{\omega_b^2 \omega_1}{2\, u^2k^2} (xY + yX) \quad . \tag{32}$$

We now assume that (x, y) are small and use the series expansion[4] for the function Z(z) to obtain, approximately

$$\frac{1}{2}(xY + yX) \doteq y^2 + \sqrt{\pi}\, (x^2 - \tfrac{1}{2})y - x^2 \quad . \tag{33}$$

Using Eqs. (33) and (32), we get

$$y^2 + 2by - x^2 = 0 \tag{34}$$

where

$$b = \frac{\sqrt{\pi}}{2}\left(x^2 - \frac{1}{2}\right) + \sqrt{2}\,\frac{u}{V_o}\left(\frac{\omega_p u}{\omega_b V_o}\right)^2 \gg 1 \quad . \tag{35}$$

Thus, we obtain

$$y = -b + \sqrt{b^2 + x^2} \simeq \frac{1}{2}\frac{x^2}{b} \quad . \tag{36}$$

With the use of the extremal value for x given by Eq. (27), the growth rate is

$$\omega_2 = \frac{1}{4}\frac{\omega_p}{\left(\frac{\omega_p u}{\omega_b V_o}\right)^2} \quad . \tag{37}$$

Provided that Eq. (37) is consistent with the initial assumption of small y, this equation gives a good estimate of the growth rate at high beam temperature. This is substantiated by the numerical work. In the event that Eq. (37) violates the assumption of small y, the high-frequency, high-temperature growth rate can be estimated by the procedure for low-temperature beams.

B. The Low-Temperature Limit

1. General

The arguments leading to the definition of a high-temperature limit, Eq. (19), provide also a basis for defining the low-temperature limit; that is, a "cold" beam is one that satisfies the condition

$$\tau \equiv \left(\frac{\omega_p u}{\omega_b V_o}\right) \ll 1 \quad . \tag{38}$$

In this case, approximate analytic solutions of the dispersion equation can be obtained that are in essential agreement with numerical results. Although these results are derived in other reports,[1-3] they are derived here, also, for completeness.

2. High-Frequency Oscillations

Consider the case that

$$\begin{aligned} &(a)\quad \omega \simeq kV_o \simeq \omega_p \\ &(b)\quad \omega_2 \ll \omega_1 \\ &(c)\quad \nu \ll \omega_1 \quad . \end{aligned} \tag{39}$$

For the case of a "cold" beam, we anticipate that $x \gg 1$ and $y \gg 1$. In this situation the asymptotic expansion for the function $Z(z)$ can be used. This is given by[4]

$$Z(z) \simeq -\frac{1}{z}\left[1 + \frac{1}{2z^2} + \ldots\right], \quad y > 0 \quad . \tag{40}$$

To this approximation

$$xY + yX = \frac{xy}{\left(x^2 + y^2\right)^2} \quad . \tag{41}$$

According to the assumptions of Eqs. (39) and (41), and using the approximate condition of Eq. (30), Eq. (12) can be rewritten

$$2\omega_2 + \nu \approx -\frac{\omega_1\omega_b^2}{u^2k^2}\,\frac{xy}{\left(x^2 + y^2\right)^2} \quad . \tag{42}$$

Using the representation $\omega - kV_o = \rho e^{i\varphi}$, Eq. (42) can be rewritten

$$\omega_2^2\left(\omega^2 + \frac{\nu}{2}\right) = -\,\omega_1\,\omega_b^2 \sin^3\varphi \cos\varphi \tag{43}$$

which has a maximum for $\tan\varphi = -\sqrt{3}$. Hence, the maximum growth rate is given by

$$\omega_2^2\left(\omega_2 + \frac{\nu}{2}\right) = \frac{3^{3/2}}{16}\,\omega_p\,\omega_b^2 \qquad (44)$$

which is easily solved in two cases:

$$\text{(a)}\quad \omega_2 \gg \nu:\quad \omega_2 = \frac{\sqrt{3}}{2}\left(\frac{\omega_p\omega_b^2}{2}\right)^{1/3} \qquad (45)$$

$$\text{(b)}\quad \nu \gg \omega_2:\quad \omega_2 = \left(\frac{3}{4}\right)^{3/4}\left(\frac{\omega_p\omega_b^2}{\nu}\right)^{1/2}\quad . \qquad (46)$$

3. <u>Low-Frequency Oscillations</u>

Assume for this case that

$$\begin{aligned}&\text{(a)}\quad \omega_1 \approx kV_o \ll \omega_p\\ &\text{(b)}\quad \omega_2 \ll \omega_1 \qquad (47)\\ &\text{(c)}\quad \nu \ll \omega_1\quad .\end{aligned}$$

Then, Eqs. (11) and (12) give, approximately,

$$\left(\frac{\omega_p u}{\omega_b V_o}\right)^2 = (1 + xX - yY) - \frac{\omega_2}{\omega_1}(xY + yX) \qquad (48)$$

$$\left(1 + \frac{\nu}{\omega_2}\right)\left(\frac{\omega_p u}{\omega_b V_o}\right)^2 = -(1 + xX - yY) - \frac{\omega_1}{\omega_2}(xY + yX) \quad . \qquad (49)$$

Eliminating the term $\left(\frac{\omega_p u}{\omega_b V_o}\right)^2$ from these equations,

$$\left(2 + \frac{\nu}{\omega_2}\right)(1 + xX - yY) = -\left(\frac{\omega_1}{\omega_2} - \frac{\omega_2}{\omega_1} - \frac{\nu}{\omega_1}\right)(xY + yX) \quad . \tag{50}$$

In view of the assumptions of Eq. (47), Eq. (50) gives approximately

$$\left(2 + \frac{\nu}{\omega_2}\right)(1 + xX - yY) \approx -\frac{\omega_1}{\omega_2}(xY + yX) \tag{51}$$

which requires $(1 + xX - yY) \gg -(xY + yX)$. (52)
Consequently, Eq. (48) can be rewritten

$$1 + xX - yY \simeq \left(\frac{\omega_p u}{\omega_b V_o}\right)^2 \lll 1 \tag{53}$$

for the low-temperature case. Now by inspection of the plotted functions in Fried and Conte,[4] it can be concluded that Eqs. (52) and (53) can be satisfied only under the condition

$$-x \ll 1, \; y \gg 1 \quad .$$

Consequently, we can use the asymptotic expansion to obtain

$$1 + xX - yY \simeq \frac{y^2 - x^2}{2\left(x^2 + y^2\right)^2} \approx \frac{1}{2y^2} \tag{54}$$

so that from Eq. (51)

$$y = \frac{1}{\sqrt{2}} \left(\frac{\omega_b V_o}{\omega_p u}\right) \tag{55a}$$

or

$$\omega_2 = \frac{\omega_b}{\omega_p} (kV_o) \quad . \tag{55b}$$

To obtain x, the asymptotic expansion

$$xY + yX \simeq \frac{xy}{\left(x^2 + y^2\right)^2} \approx \frac{x}{y^3} \tag{56}$$

can be used in conjunction with Eqs. (49) and (53), which give

$$(xY + yX) = - \left(2 + \frac{\nu}{\omega_2}\right) \left(\frac{\omega_p u}{\omega_b V_o}\right)^2 \frac{\omega_2}{\omega_1} \quad . \tag{57}$$

Then from Eqs. (55), (56), and (57) are obtained the approximate results

$$\omega_2 \gg \nu, \; x = - \frac{\omega_b}{\omega_p} y \tag{58}$$

$$\nu \gg \omega_2, \; x = - \frac{\nu}{2(kV_o)} y \quad . \tag{59}$$

C. The Intermediate-Temperature Beam

Consistent with previous arguments, the intermediate temperature beam can be defined for the case that

$$\tau \equiv \left(\frac{\omega_p}{\omega_b}\frac{u}{V_o}\right) \approx 1 \quad . \tag{60}$$

Analysis of the dispersion equation in this situation is, in general, more difficult than for the cases of high- or low-beam temperature. We will discuss some cases where reasonably good analytic approximations are obtained.

1. High-Frequency Waves ($\omega_1 \approx \omega_p$)

For the case of a weak beam ($\omega_p >> \omega_b$), the cold-beam solution holds. To show this, from Eq. (45) is obtained

$$y \approx 0.49 \left[\frac{V_o/u}{\left(\frac{\omega_p^2 u^2}{\omega_b^2 V_o^2}\right)}\right]^{1/3} \quad . \tag{61}$$

which is greater than unity for the weak-beam assumption, provided that Eq. (60) holds. Hence, the asymptotic expansion used to derive Eq. (45) still holds, so that for the intermediate-temperature beam, the high frequency solutions are the same as for the low-temperature beam:

$$\text{(a)} \quad \omega_2 >> \nu: \quad \omega_2 = \frac{\sqrt{3}}{2}\left(\frac{\omega_p \omega_b^2}{2}\right)^{1/3} \tag{45'}$$

$$\text{(b)} \quad \nu >> \nu_2: \quad \omega_2 = \left(\frac{3}{4}\right)^{3/4}\left(\frac{\omega_p \omega_b^2}{\nu}\right)^{1/2} \quad . \tag{46'}$$

2. Low-Frequency Waves ($\omega_1 << \omega_p$)

Low-frequency wave solutions can be obtained for the special case

$$\frac{\omega_p}{\omega_b}\frac{u}{V_o} \lesssim 1 \quad . \tag{62}$$

With the following assumptions,

$$\begin{aligned}
&\text{(a)} \quad \omega_1 \ll \omega_p \\
&\text{(b)} \quad \omega_2, \ \nu \ll \omega_1 \\
&\text{(c)} \quad \omega_1 \approx kV_o
\end{aligned} \tag{63}$$

the arguments leading from Eq. (47) to Eq. (52) can be repeated for this case, which lead to the approximate results

$$1 + xX - yY \gg -(xX + yY) \tag{64}$$

and

$$1 + xX - yY = \left(\frac{\omega_p}{\omega_b}\frac{u}{V_o}\right)^2 \lesssim 1 \quad . \tag{65}$$

On examination* of the functions $(1 + xX - yY)$ and $(xY + yX)$, it is noted that (for $y > 0$) Eqs. (64) and (65) can be satisfied only under the conditions

$$-x \ll y < 1 \tag{66}$$

and under these conditions a series expansion gives the approximate relation

---

*See, for example, the tables and plotted results of Ref. 4.

$$1 + xX - yY \approx 1 - \sqrt{\pi}\, y + 2y^2 \quad . \tag{67}$$

Using this expansion in Eq. (65), we obtain

$$y = \frac{\sqrt{\pi}}{4}\left\{1 - \left[1 - \frac{8}{\pi}\left(1 - \frac{u^2}{V_o^2}\frac{\omega_p^2}{\omega_b^2}\right)\right]^{\frac{1}{2}}\right\} \quad . \tag{68}$$

For the numerical work, an estimate of x is also required. For $\nu \ll \omega_2$, Eqs. (49) and (65) give

$$xY + yX \approx -\, 2\,\frac{\omega_2}{\omega_1}\left(\frac{\omega_p u}{\omega_b V_o}\right)^2 \quad . \tag{69}$$

Expanding the left-hand side of Eq. (69) in a power series gives, approximately,

$$xY + yX = x(\sqrt{\pi} - 4y) \tag{70}$$

so that

$$x = -\, 2\,\frac{\omega_2}{\omega_1}\,\frac{\left(\frac{\omega_p u}{\omega_b V_o}\right)^2}{\sqrt{\pi} - 4y} \quad . \tag{71}$$

Similarly, for the case $\nu \gg \omega_2$, one obtains

$$x = -\,\frac{\frac{\nu}{\omega_1}\left(\frac{\omega_p u}{\omega_b V_o}\right)^2}{\sqrt{\pi} - 4y} \quad . \tag{72}$$

The numerical results indicate that the estimates of Eqs. (68), and (71) or (72) give close first guesses for the cases

$$\frac{\omega_p}{\omega_b}\frac{u}{V_o} = 0.9,\ 0.99 \quad . \tag{73}$$

On the other hand, these estimates do not hold for the case

$$\frac{\omega_p}{\omega_b}\frac{u}{V_o} = 1 \quad . \tag{74}$$

The numerical results obtained for this last case were obtained by trial and error.

D. Summary of Approximate Solutions

In Table I are summarized the approximate solutions obtained in the previous sections.

Table I

TABLE OF APPROXIMATE SOLUTIONS OF THE DISPERSION EQUATION

$$\tau = \frac{\omega_p u}{\omega_b V_o}$$

Beam Temperature	ω_1	x	$\omega_2 = \sqrt{2}\,(kV_o)\left(\frac{u}{V_o}\right) y$	Comments
	$\omega_1 \approx \omega_p$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{4}\,\omega_p\,\tau^{-2}$	$\nu \ll -\sqrt{\frac{\pi}{2e}}\,\frac{\omega_p}{\tau^2}$
$\tau \gg 1$	$\omega_1 \approx \omega_p$	$-\frac{1}{\sqrt{2}}$	$\omega_2 \leq 0$	$\nu \geq -\sqrt{\frac{\pi}{2e}}\,\frac{\omega_p}{\tau^2}$
	$\omega_1 \ll \omega_p$	--	$\omega_2 < 0$	
	$\omega_1 \approx \omega_p$	$-y/\sqrt{3}$	$\sqrt{3}\,\omega_p\left(\frac{\omega_b}{4\omega_p}\right)^{2/3}$	$\omega_2 \gg \nu$
	$\omega_1 \approx \omega_p$	$-y/\sqrt{3}$	$\left(\frac{3}{4}\right)^{3/4}\,\omega_p\left(\frac{\omega_b^2}{\omega_p \nu}\right)^{1/2}$	$\nu \gg \omega_2$
$\tau \ll 1$		$-\frac{\omega_b}{\omega_p}\,y$	$\frac{\omega_b}{\omega_p}\,(kV_o)$	$\nu \ll \omega_2$
	$\omega_1 \ll \omega_p$	$-\frac{\nu}{2(kV_o)}\,y$	$\frac{\omega_b}{\omega_p}\,(kV_o)$	$\omega_2 \ll \nu \ll \omega_1$
	$\omega_1 \approx \omega_p$	$-y/\sqrt{3}$	$\sqrt{3}\,\omega_p\left(\frac{\omega_b}{4\omega_p}\right)^{2/3}$	$\omega_2 \gg \nu$
	$\omega_1 \approx \omega_p$	$-y/\sqrt{3}$	$\left(\frac{3}{4}\right)^{3/4}\,\omega_p\left(\frac{\omega_b^2}{\omega_p \nu}\right)^{1/2}$	$\nu \gg \omega_2$
$\tau \lesssim 1$	$\omega_1 \ll \omega_p$	$-2\sqrt{2}\,\frac{u}{V_o}\,\frac{\tau^2 y}{(\sqrt{\pi} - 4y)}$	$\sqrt{\frac{\pi}{8}}\,(kV_o)\left(\frac{u}{V_o}\right)\left\{1 - \sqrt{1 - \frac{8}{\pi}[1 - \tau^2]}\right\}$	$\omega_2 \gg \nu$
	$\omega_1 \ll \omega_p$	$-\frac{\nu}{kV_o}\,\frac{\tau^2}{(\sqrt{\pi} - 4y)}$		$\nu \gg \omega_2$

IV NUMERICAL PROGRAM*

A. Method

The numerical results presented in this report were obtained by an application of the Method of False Position[5] (Regula Falsi) to the complex plane. The only justification offered for this approach is that it was the most successful of a number of approaches tried. In earlier work a Newton's method was employed with limited success, the success depending to a large extent upon where in the complex z-plane the solution lay.

The Method of False Position can be described briefly as follows: for a given set of beam-plasma parameters, Eq. (8) can be written

$$f(z) = 1 - \frac{\omega_p^2}{\omega(\omega + i\nu)} - \frac{\omega_b^2}{u^2 k^2} [1 + zZ] \tag{75}$$

where we seek a z such that

$$f(z) = 0 \quad . \tag{76}$$

The procedure is to make an initial guess z_o, and determine $z_1 = z_o(1 + \epsilon)$, where $\epsilon \ll 1$. We then calculate the sequence z_n, for $n = 1, 2, \ldots$, according to the relation

$$z_{n+1} = z_o - \frac{f(z_o)\ [z_o - z_n]}{f(z_o) - f(z_n)} \quad . \tag{77}$$

---

*The method of solution and the programming of this problem are due to the efforts of Mr. Len McCulley of the Mathematical Sciences Department. The program was written in BALGOL for a Burroughs 220 computer.

The sequence is stopped when the absolute value $|z_{n+1} - z_n|$ is less than some predetermined small value. The function $Z(z_n)$ is calculated along the way.

B. Numerical Results

Machine calculations for the growth rate were carried out for several values of plasma and beam plasma frequency as shown:

ω_p	10^{12}	10^{11}	10^{12}	10^{11}
ω_b	10^{9}	10^{8}	10^{8}	10^{9}

For each case, the calculations were carried out for beam temperatures as shown:

$\tau \equiv \frac{\omega_p u}{\omega_b V_o}$	0.1	0.9	0.99	1.0	10.0

The results of these calculations are shown in Figs. 1-4, where the growth rate of the instability is plotted as a function of $kV_o \approx \omega_1$.

The points connected by solid lines in Figs. 1-4 are obtained for a collision frequency $\nu = 10^5$. Results obtained for collision frequencies 10^7 and 10^9 are essentially the same as for the case $\nu = 10^5$ except at lower frequencies where the growth rate tends to be enhanced by collisions. These results are shown roughly by the dashed branch lines in Figs. 1-4.

For high beam temperature, $\tau = 10$, the growth rate is found to have a maximum in the vicinity of the plasma frequency, ω_p. The maximum growth rate as determined numerically was found to be in close agreement with the theoretical results of Eq. (37). It should be noted that the maximum growth rate for the high-temperature case is not significantly less than the low-temperature growth rate for any of the cases considered here. For wave frequencies slightly greater or less than the maximum, the growth rate drops quite rapidly to a condition of damped waves. These results are indicated by a solid vertical line in Figs. 1-4.

Numerical results for the effect of collisional damping in the high temperature limit are shown below. Here, ν^* is the minimum value of collision frequency for which wave damping is obtained. These results are obtained for the condition $\tau = 10$, as follows:

ω_p, ω_b	10^{12}, 10^9	10^{11}, 10^8	10^{12}, 10^8	10^{11}, 10^9
$\frac{\tau^2}{\omega_p} \nu^*$	0.73	0.73	0.71	0.61

From the approximate theory leading to Eq. (28) is obtained

$$\frac{\tau^2}{\omega_p} \nu^* = 0.76 \quad .$$

V CONCLUDING REMARKS

If stability with respect to electrostatic waves is defined in terms of the high-temperature limit, $\tau \gg 1$, one sees two opposing influences when comparing relativistic and non-relativistic beams. That is, since the high-temperature limit is defined by

$$\tau = \frac{\omega_p u}{\omega_b V_o} \gg 1$$

where

$$\frac{\omega_p}{\omega_b} = \left(\frac{n_p}{n_b}\right)^{\frac{1}{2}} \gamma^{3/2}$$

then the condition for stability can be written

$$\tau = \left(\frac{n_p}{n_b}\right)^{\frac{1}{2}} \left(\frac{u}{V_o}\right) \gamma^{3/2} \gg 1 \quad .$$

It is evident from the above inequality that, for given plasma and beam particle density, the product of the beam velocity spread u and the relativistic factor $\gamma^{3/2}$ is important in satisfying the inequality. Thus in going to relativistic velocities, it becomes difficult to achieve a large velocity spread, although this is counteracted to some extent by the larger relativistic factor.

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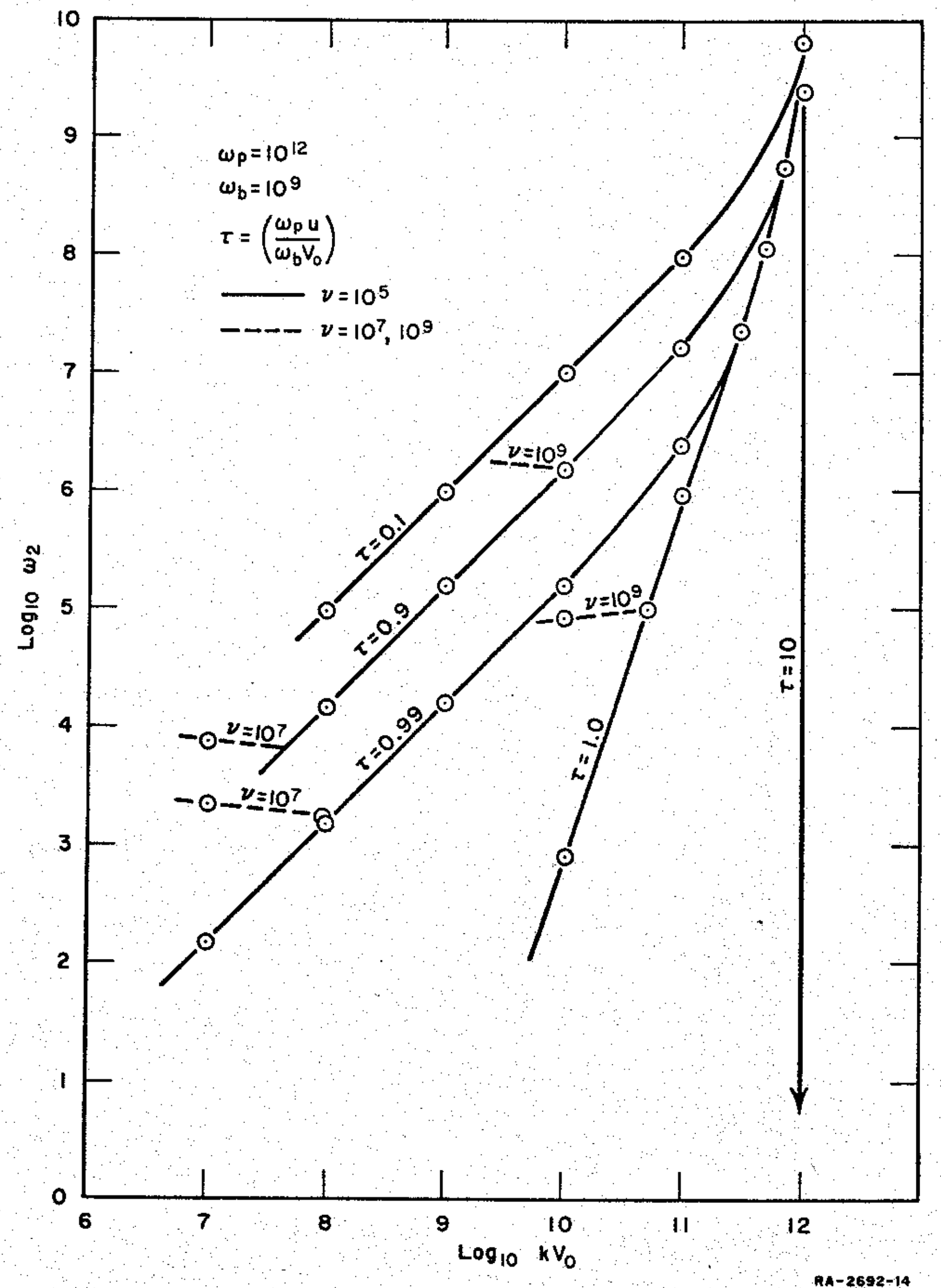

FIG. 1 GROWTH RATE OF ELECTROSTATIC INSTABILITY

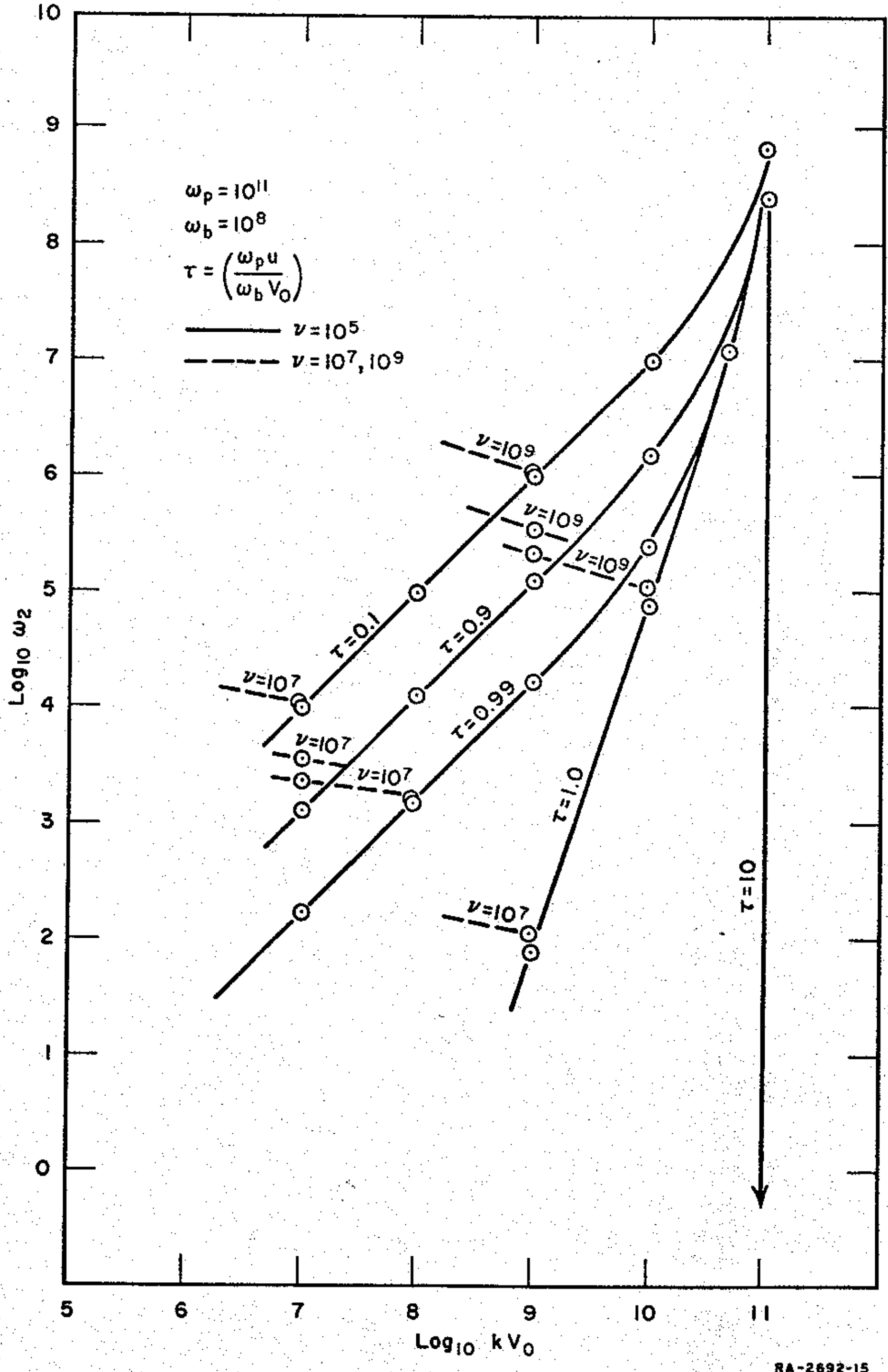

FIG. 2 GROWTH RATE OF ELECTROSTATIC INSTABILITY

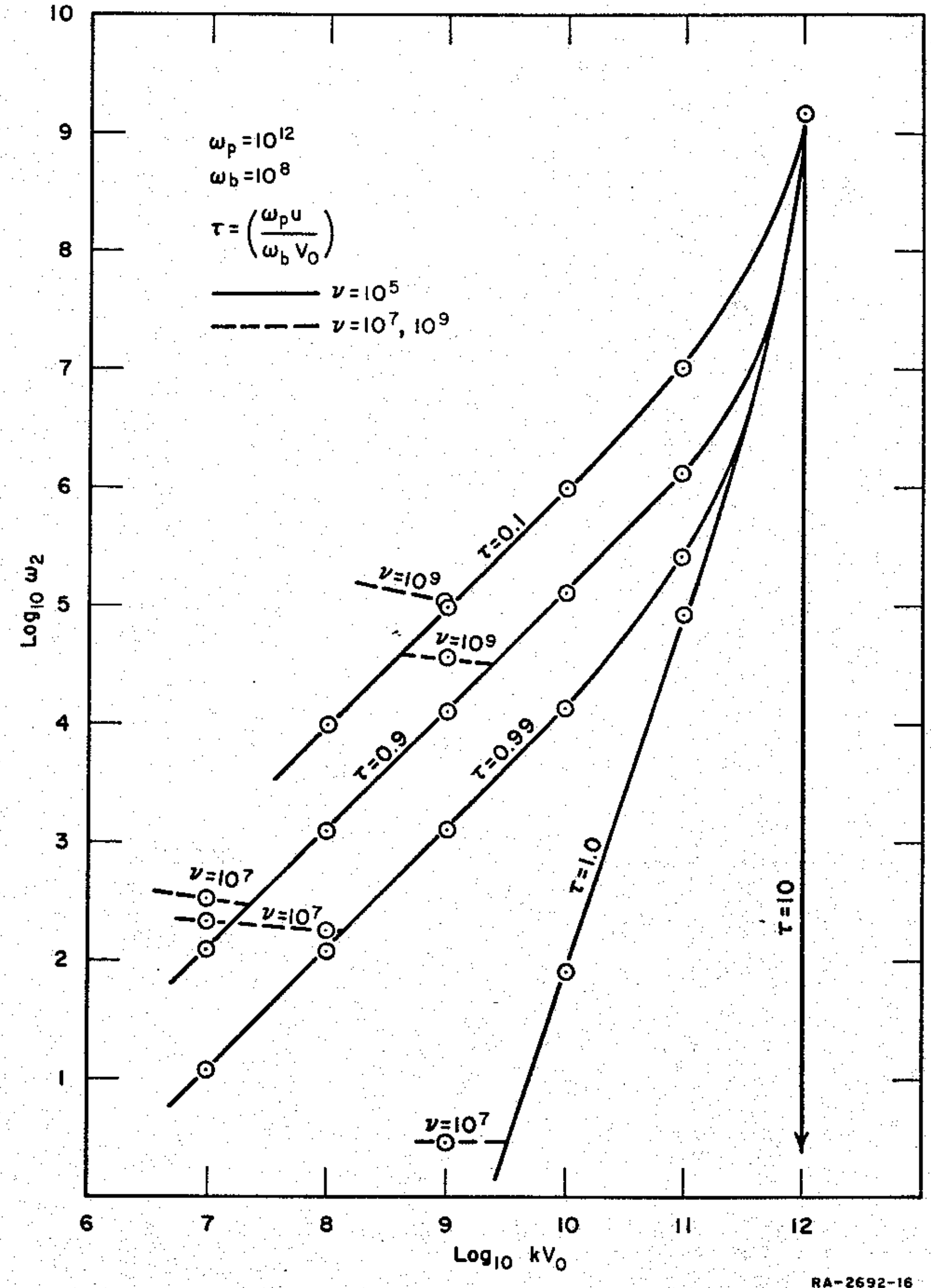

FIG. 3 GROWTH RATE OF ELECTROSTATIC INSTABILITY

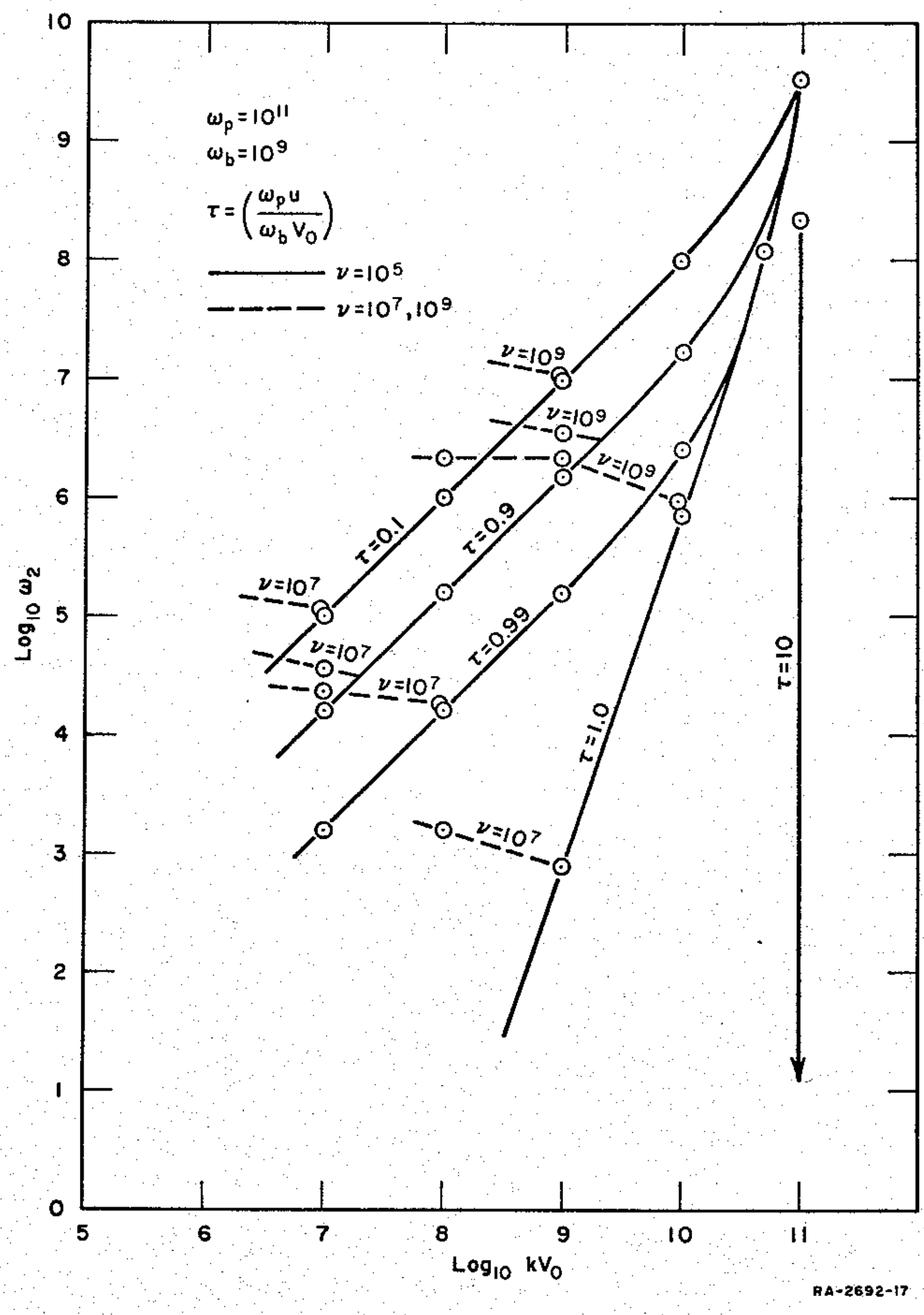

FIG. 4 GROWTH RATE OF ELECTROSTATIC INSTABILITY